

Rough dependence upon initial data exemplified by explicit solutions and the effect of viscosity

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ABSTRACT. In this article, we present some explicit solutions showing rough dependence upon initial data. We also studies viscous effects. An extension of a theorem of Cauchy to the viscous case is also presented.

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1. Introduction

It is well-known that both Navier-Stokes and Euler equations are locally well-posed in the Sobolev space H^s ($s > \frac{d}{2} + 1$) where d is the spatial dimension [4] [5]. In two dimensions ($d = 2$), the well-posedness is also global. The solutions are in the space $C^0([0, T], H^3)$, where T is either finite or infinite. The solution operator F^t is a one-parameter family of maps on H^3 . For each fixed t , F^t maps the initial condition $u(0)$ to the solution's value at time t , $u(t)$. F^t is continuous in $u(0)$. In [1], a simple example was presented showing that the solution operator F^t is not uniformly continuous in $u(0)$ under the Euler dynamics. In [2] [3], the nowhere differentiability of F^t under the Euler dynamics was proved. In [7], we derived the upper bound on the derivative of the solution operator under the Navier-Stokes dynamics: $\|\nabla F^t\| \leq e^{\sigma\sqrt{Re}\sqrt{t} + \sigma_1 t}$ where σ and σ_1 are constants depending only on the norm of the base solution. In this article, we present explicit solutions to both 2D Euler and 2D Navier-Stokes equations. These solutions show that under the Euler dynamics, $\|\nabla F^t\| = \infty$, and under the Navier-Stokes dynamics,

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$\|\nabla F^t\| \rightarrow \infty$ as the Reynolds number approaches infinity. We name the above nature of dependence of the solution operator on initial data as rough dependence on initial data.

2. Explicit solutions with rough dependence on initial data

In this section, we present explicit solutions to the 2D Euler equations and 2D Navier-Stokes equations, which show rough dependence on initial data. Consider the following 2D Euler equations

$$(2.1) \quad \partial_t u + u \cdot \nabla u = -\nabla p, \quad \nabla \cdot u = 0,$$

and 2D Navier-Stokes equations

$$(2.2) \quad \partial_t u + u \cdot \nabla u = -\nabla p + \frac{1}{Re} \Delta u, \quad \nabla \cdot u = 0,$$

under periodic boundary condition with period domain $[0, 2\pi] \times [0, 2\pi]$, where $u = (u_1, u_2)$ is the velocity, p is pressure, and the spatial coordinate is denoted by $x = (x_1, x_2)$. First we study the following simple explicit solution to the 2D Euler equations,

$$(2.3) \quad u_1 = \sum_{n=1}^{\infty} \frac{1}{n^{3+\gamma}} \sin[n(x_2 - \sigma t)], \quad u_2 = \sigma,$$

where $\frac{1}{2} < \gamma \leq 1$, and σ is a real parameter. We view this solution as a solution in the space $C^0([0, \infty), H^3)$, where H^3 is the Sobolev space. We select the Sobolev space H^3 because both the 2D Euler equations and the 2D Navier-Stokes equations are well-posed in H^3 . The same construction in this article can be easily extended to any Sobolev space H^s ($s \in \mathbb{R}$). Denote by F^t the solution operator of either the 2D Euler equations or the 2D Navier-Stokes equations. For any fixed t , one can view F^t as a map in H^3 ,

$$F^t : u(0) \rightarrow u(t).$$

The initial condition of the solution (2.3) is

$$u_1 = \sum_{n=1}^{\infty} \frac{1}{n^{3+\gamma}} \sin(nx_2), \quad u_2 = \sigma.$$

By varying σ , we get a variation direction of the initial condition,

$$du_1(0) = 0, \quad du_2(0) = d\sigma,$$

which leads to the directional derivative $\partial_\sigma F^t$ of F^t :

$$\frac{\partial u_1}{\partial \sigma} = \sum_{n=1}^{\infty} \frac{-t}{n^{2+\gamma}} \cos[n(x_2 - \sigma t)], \quad \frac{\partial u_2}{\partial \sigma} = 1.$$

The norm of $\partial_\sigma F^t$ is given by

$$\begin{aligned} \|\partial_\sigma F^t\|_{H^3}^2 &= 4\pi^2 + 2\pi^2 t^2 \sum_{n=1}^{\infty} (1 + n^2 + n^4 + n^6) \frac{1}{n^{4+2\gamma}} \\ &= 4\pi^2 + 2\pi^2 t^2 \sum_{n=1}^{\infty} \left(\frac{1}{n^{4+2\gamma}} + \frac{1}{n^{2+2\gamma}} + \frac{1}{n^{2\gamma}} + n^{2-2\gamma} \right) \\ &= \infty, \quad (t > 0), \end{aligned}$$

where the last series is divergent when $\frac{1}{2} < \gamma \leq 1$. Thus the directional derivative $\partial_\sigma F^t$ does not exist. Therefore the derivative ∇F^t does not exist in view of the fact that the norm of the derivative ∇F^t is greater than or equal to the norm of the directional derivative $\partial_\sigma F^t$. In fact, the solution operator F^t is nowhere differentiable [2].

The corresponding solution of (2.3) to the 2D Navier-Stokes equations (2.2) is

$$(2.4) \quad u_1 = \sum_{n=1}^{\infty} \frac{1}{n^{3+\gamma}} e^{-\frac{n^2}{Re}t} \sin[n(x_2 - \sigma t)], \quad u_2 = \sigma,$$

which is also a solution in the space $C^0([0, \infty), H^3)$. The directional derivative $\partial_\sigma F^t$ of F^t is

$$\frac{\partial u_1}{\partial \sigma} = \sum_{n=1}^{\infty} \frac{-t}{n^{2+\gamma}} e^{-\frac{n^2}{Re}t} \cos[n(x_2 - \sigma t)], \quad \frac{\partial u_2}{\partial \sigma} = 1.$$

The norm of $\partial_\sigma F^t$ is given by

$$\|\partial_\sigma F^t\|_{H^3}^2 = 4\pi^2 + 2\pi^2 t^2 \sum_{n=1}^{\infty} (1 + n^2 + n^4 + n^6) \frac{n^2}{n^{6+2\gamma}} e^{-2\frac{n^2}{Re}t}.$$

Let

$$(2.5) \quad g(\xi) = t^2 \xi e^{-\frac{2t}{Re}\xi}.$$

The maximum of $g(\xi)$ is given by

$$g'(\xi) = t^2 e^{-\frac{2t}{Re}\xi} \left(1 - \frac{2t}{Re}\xi\right) = 0,$$

that is,

$$(2.6) \quad \xi = \frac{Re}{2t},$$

where the maximal value of g is

$$g = \frac{t}{2} \frac{Re}{e} e^{-1}.$$

Thus

$$\|\partial_\sigma F^t\|_{H^3} \leq \left(\frac{1}{\sigma} + \sqrt{Re}\sqrt{t}\sqrt{\frac{1}{2e}}\right) \|u(0)\|_{H^3}.$$

Notice that for the full derivative ∇F^t , the upper bound is given as [7],

$$\|\nabla F^t\|_{H^3 \rightarrow H^3} \leq e^{\sigma\sqrt{Re}\sqrt{t} + \sigma_1 t},$$

where σ and σ_1 are two constants depending on the H^3 norm of the base solution.

From (2.6), let

$$n = \left\lceil \sqrt{\frac{Re}{2t}} \right\rceil, \quad \left(\text{the integer part of } \sqrt{\frac{Re}{2t}} \right),$$

then

$$\frac{1}{2} \sqrt{\frac{Re}{2t}} < n \leq \sqrt{\frac{Re}{2t}}, \quad \text{when } \sqrt{\frac{Re}{2t}} \geq 1.$$

We have

$$\begin{aligned}
& \|\partial_\sigma F^t\|_{H^3}^2 > 4\pi^2 + \\
& 2\pi^2 t^2 (1 + n^2 + n^4 + n^6) \frac{n^2}{n^{6+2\gamma}} e^{-2\frac{n^2}{Re}t} \\
& > 4\pi^2 + 2\pi^2 t^2 n^{2-2\gamma} e^{-2\frac{n^2}{Re}t} \\
& \geq 4\pi^2 + 2\pi^2 t^2 \left(\frac{1}{2} \sqrt{\frac{Re}{2t}} \right)^{2-2\gamma} e^{-1} \\
& = (2\pi)^2 + \left[\frac{\sqrt{2}}{\sqrt{e}} \pi t^\gamma \left(\frac{\sqrt{t}\sqrt{Re}}{2\sqrt{2}} \right)^{1-\gamma} \right]^2 \\
& \geq \frac{1}{2} \left[2\pi + \frac{\sqrt{2}}{\sqrt{e}} \pi t^\gamma \left(\frac{\sqrt{t}\sqrt{Re}}{2\sqrt{2}} \right)^{1-\gamma} \right]^2.
\end{aligned}$$

Thus

$$\|\partial_\sigma F^t\|_{H^3} > \sqrt{2}\pi + \frac{\pi}{\sqrt{e}} t^\gamma \left(\frac{\sqrt{t}\sqrt{Re}}{2\sqrt{2}} \right)^{1-\gamma}.$$

As $Re \rightarrow \infty$,

$$\|\partial_\sigma F^t\|_{H^3} \rightarrow \infty,$$

thus

$$\|\nabla F^t\|_{H^3} \rightarrow \infty.$$

From another perspective, one can also obtain a lower bound for $\|\partial_\sigma F^t\|_{H^3}$. Fix a $t > 0$, for each n , the maximum (2.6) specifies a Reynolds number $Re^{(n)}$ (with $\xi = n^2$),

$$Re^{(n)} = 2tn^2,$$

where

$$g(n^2) = e^{-1} t^2 n^2.$$

Since

$$\begin{aligned}
& \|\partial_\sigma F^t\|_{H^3}^2 > 4\pi^2 + \\
& 2\pi^2 t^2 (1 + n^2 + n^4 + n^6) \frac{n^2}{n^{6+2\gamma}} e^{-2\frac{n^2}{Re}t} \\
& > 4\pi^2 + 2\pi^2 t^2 n^{2-2\gamma} e^{-2\frac{n^2}{Re}t},
\end{aligned}$$

we have

$$\|\partial_\sigma F^t\|_{H^3} > 2\pi \sqrt{1 + \frac{t^2}{2e} n^{2-2\gamma}}, \text{ when } Re = Re^{(n)}.$$

Therefore, as $Re^{(n)} \rightarrow \infty$ ($n \rightarrow \infty$),

$$\|\partial_\sigma F^t\|_{H^3} \rightarrow \infty.$$

More general solutions with the same property of rough dependence on initial data can be derived as follows. We expand the vorticity into a Fourier series

$$\omega = \sum_{k \in \mathbb{Z}^2 / \{0\}} \omega_k e^{ik \cdot x}$$

where

$$\omega_{-k} = \overline{\omega_k}, \quad k = (k_1, k_2), \quad x = (x_1, x_2).$$

When the velocity has zero spatial mean, the 2D Euler equations can be re-written as

$$\dot{\omega}_k = \sum_{k=p+q} A(p, q) \omega_p \omega_q$$

where

$$(2.7) \quad A(p, q) = \frac{1}{2}(|q|^{-2} - |p|^{-2}) \begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix}.$$

When $p \parallel q$,

$$A(p, q) = 0.$$

This implies that for any $k \in \mathbb{Z}^2 / \{0\}$,

$$(2.8) \quad \omega = \sum_{n \in \mathbb{Z} / \{0\}} C_n e^{in(k \cdot x)}$$

is a steady solution to the 2D Euler equation (in vorticity form) [6], where C_n 's are complex constants, and $C_{-n} = \overline{C_n}$. In terms of velocity, this solution has the form

$$\begin{aligned} u_1 &= \sum_{n \in \mathbb{Z} / \{0\}} \frac{ik_2 C_n}{n|k|^2} e^{in(k \cdot x)}, \\ u_2 &= - \sum_{n \in \mathbb{Z} / \{0\}} \frac{ik_1 C_n}{n|k|^2} e^{in(k \cdot x)}. \end{aligned}$$

Under a translation in velocity, this solution is transformed into the following time-dependent solution,

$$(2.9) \quad \begin{aligned} u_1 &= \sigma_1 + \sum_{n \in \mathbb{Z} / \{0\}} \frac{ik_2 C_n}{n|k|^2} e^{in[k_1(x_1 - \sigma_1 t) + k_2(x_2 - \sigma_2 t)]}, \\ u_2 &= \sigma_2 - \sum_{n \in \mathbb{Z} / \{0\}} \frac{ik_1 C_n}{n|k|^2} e^{in[k_1(x_1 - \sigma_1 t) + k_2(x_2 - \sigma_2 t)]}, \end{aligned}$$

where σ_1 and σ_2 are real constants. By choosing

$$(2.10) \quad C_n = \frac{1}{n^{2+\gamma}}, \quad \left(\frac{1}{2} < \gamma \leq 1\right),$$

we get a solution which has the same property of rough dependence on initial data as the solution (2.3). The initial condition of the solution is

$$\begin{aligned} u_1(0) &= \sigma_1 + \sum_{n \in \mathbb{Z} / \{0\}} \frac{ik_2}{n^{3+\gamma}|k|^2} e^{in(k_1 x_1 + k_2 x_2)}, \\ u_2(0) &= \sigma_2 - \sum_{n \in \mathbb{Z} / \{0\}} \frac{ik_1}{n^{3+\gamma}|k|^2} e^{in(k_1 x_1 + k_2 x_2)}. \end{aligned}$$

By varying σ_1 (or σ_2), we get a variation direction

$$du_1(0) = d\sigma_1, \quad du_2(0) = 0,$$

and the corresponding directional derivative $\partial_{\sigma_1} F^t$ of the solution operator F^t is

$$\begin{aligned}\frac{\partial u_1}{\partial \sigma_1} &= 1 + \sum_{n \in \mathbb{Z}/\{0\}} \frac{t}{n^{2+\gamma}} \frac{k_1 k_2}{|k|^2} e^{in[k_1(x_1 - \sigma_1 t) + k_2(x_2 - \sigma_2 t)]}, \\ \frac{\partial u_2}{\partial \sigma_1} &= - \sum_{n \in \mathbb{Z}/\{0\}} \frac{t}{n^{2+\gamma}} \frac{k_1^2}{|k|^2} e^{in[k_1(x_1 - \sigma_1 t) + k_2(x_2 - \sigma_2 t)]}.\end{aligned}$$

The norm of $\partial_{\sigma_1} F^t$ is given by

$$\begin{aligned}\|\partial_{\sigma_1} F^t\|_{H^3} &= 4\pi^2 + 4\pi^2 t^2 \frac{k_1^2}{|k|^2} \sum_{n \in \mathbb{Z}/\{0\}} \left(\right. \\ &\quad \left. 1 + n^2 |k|^2 + n^4 |k|^4 + n^6 |k|^6 \right) \frac{1}{n^{4+2\gamma}} = \infty,\end{aligned}$$

when $t > 0$ and $k_1 \neq 0$.

Under viscous effect, the corresponding solution of (2.9) to the 2D Navier-Stokes equations (2.2) is given by

$$\begin{aligned}(2.11) \quad u_1 &= \sigma_1 + \sum_{n \in \mathbb{Z}/\{0\}} \frac{ik_2 C_n}{n|k|^2} e^{-\frac{n^2 |k|^2}{Re} t} e^{in[k_1(x_1 - \sigma_1 t) + k_2(x_2 - \sigma_2 t)]}, \\ u_2 &= \sigma_2 - \sum_{n \in \mathbb{Z}/\{0\}} \frac{ik_1 C_n}{n|k|^2} e^{-\frac{n^2 |k|^2}{Re} t} e^{in[k_1(x_1 - \sigma_1 t) + k_2(x_2 - \sigma_2 t)]}.\end{aligned}$$

We still choose C_n as in (2.10). The directional derivative $\partial_{\sigma_1} F^t$ is given by

$$\begin{aligned}\frac{\partial u_1}{\partial \sigma_1} &= 1 + \sum_{n \in \mathbb{Z}/\{0\}} \frac{t}{n^{2+\gamma}} \frac{k_1 k_2}{|k|^2} e^{-\frac{n^2 |k|^2}{Re} t} e^{in[k_1(x_1 - \sigma_1 t) + k_2(x_2 - \sigma_2 t)]}, \\ \frac{\partial u_2}{\partial \sigma_1} &= - \sum_{n \in \mathbb{Z}/\{0\}} \frac{t}{n^{2+\gamma}} \frac{k_1^2}{|k|^2} e^{-\frac{n^2 |k|^2}{Re} t} e^{in[k_1(x_1 - \sigma_1 t) + k_2(x_2 - \sigma_2 t)]}.\end{aligned}$$

The norm of $\partial_{\sigma_1} F^t$ is given by

$$\begin{aligned}\|\partial_{\sigma_1} F^t\|_{H^3} &= 4\pi^2 + 4\pi^2 t^2 \frac{k_1^2}{|k|^2} \sum_{n \in \mathbb{Z}/\{0\}} \left(\right. \\ &\quad \left. 1 + n^2 |k|^2 + n^4 |k|^4 + n^6 |k|^6 \right) \frac{n^2}{n^{6+2\gamma}} e^{-2\frac{n^2 |k|^2}{Re} t}.\end{aligned}$$

By the result on the function $g(\xi)$ (2.5) with $\xi = n^2 |k|^2$, we get

$$\|\partial_{\sigma_1} F^t\|_{H^3} \leq \left(\frac{1}{\sqrt{\sigma_1^2 + \sigma_2^2}} + \frac{|k_1|}{|k|} \sqrt{Re} \sqrt{t} \sqrt{\frac{1}{2e}} \right) \|u(0)\|_{H^3}.$$

From (2.6), let

$$n = \left\lceil \frac{1}{|k|} \sqrt{\frac{Re}{2t}} \right\rceil, \quad \left(\text{the integer part of } \frac{1}{|k|} \sqrt{\frac{Re}{2t}} \right),$$

then

$$\frac{1}{2} \frac{1}{|k|} \sqrt{\frac{Re}{2t}} < n \leq \frac{1}{|k|} \sqrt{\frac{Re}{2t}}, \quad \text{when } \frac{1}{|k|} \sqrt{\frac{Re}{2t}} \geq 1.$$

We have

$$\begin{aligned}
& \|\partial_{\sigma_1} F^t\|_{H^3}^2 > 4\pi^2 + \\
& 4\pi^2 t^2 \frac{k_1^2}{|k|^2} (1 + n^2 |k|^2 + n^4 |k|^4 + n^6 |k|^6) \frac{n^2}{n^{6+2\gamma}} e^{-2\frac{n^2 |k|^2}{Re} t} \\
& \geq 4\pi^2 + 4\pi^2 t^2 k_1^2 |k|^4 n^{2-2\gamma} e^{-2\frac{n^2 |k|^2}{Re} t} \\
& \geq 4\pi^2 + 4\pi^2 t^2 k_1^2 |k|^4 \left(\frac{1}{2} \frac{1}{|k|} \sqrt{\frac{Re}{2t}} \right)^{2-2\gamma} e^{-1} \\
& = (2\pi)^2 + \left[\frac{2\pi}{\sqrt{e}} k_1 |k|^{1+\gamma} t^\gamma \left(\frac{\sqrt{t} \sqrt{Re}}{2\sqrt{2}} \right)^{1-\gamma} \right]^2 \\
& \geq \frac{1}{2} \left[2\pi + \frac{2\pi}{\sqrt{e}} k_1 |k|^{1+\gamma} t^\gamma \left(\frac{\sqrt{t} \sqrt{Re}}{2\sqrt{2}} \right)^{1-\gamma} \right]^2.
\end{aligned}$$

Thus

$$\|\partial_{\sigma_1} F^t\|_{H^3} > \sqrt{2}\pi + \frac{\sqrt{2}\pi}{\sqrt{e}} k_1 |k|^{1+\gamma} t^\gamma \left(\frac{\sqrt{t} \sqrt{Re}}{2\sqrt{2}} \right)^{1-\gamma}.$$

As $Re \rightarrow \infty$,

$$\|\partial_{\sigma_1} F^t\|_{H^3} \rightarrow \infty,$$

thus

$$\|\nabla F^t\|_{H^3} \rightarrow \infty.$$

Now we go back to the formula (2.7), when $|p| = |q|$,

$$A(p, q) = 0.$$

This implies that

$$\omega = \sum_{|k|=c} C_k e^{ik \cdot x}, \quad (c \text{ is a constant})$$

is a steady solution of the 2D Euler equations (in vorticity form) [6], where C_k 's are complex constants, and $C_{-k} = \overline{C_k}$. In terms of velocity, this solution has the form

$$\begin{aligned}
u_1 &= \sum_{|k|=c} \frac{ik_2 C_k}{|k|^2} e^{ik \cdot x}, \\
u_2 &= - \sum_{|k|=c} \frac{ik_1 C_k}{|k|^2} e^{ik \cdot x}.
\end{aligned}$$

Under a translation in velocity, this solution is transformed into the following time-dependent solution,

$$\begin{aligned}
u_1 &= \sigma_1 + \sum_{|k|=c} \frac{ik_2 C_k}{|k|^2} e^{i[k_1(x_1 - \sigma_1 t) + k_2(x_2 - \sigma_2 t)]}, \\
u_2 &= \sigma_2 - \sum_{|k|=c} \frac{ik_1 C_k}{|k|^2} e^{i[k_1(x_1 - \sigma_1 t) + k_2(x_2 - \sigma_2 t)]},
\end{aligned} \tag{2.12}$$

where σ_1 and σ_2 are real constants. A special case of (2.12) was given in [1]. As shown in [1], by choosing σ_1 (and/or σ_2) to be $\frac{1}{n}$, k_1 (and/or k_2) to be n , and C_k to be n^{1+s} ($s \in \mathbb{R}$), we end up with a sequence of solutions. As $n \rightarrow \infty$, this sequence of solutions shows that the solution operator is not uniformly continuous in H^s .

3. Extension of a theorem of Cauchy to the viscous case

Let $\xi(x, t)$ be the fluid particle trajectory starting from x at the initial time $t = 0$,

$$(3.1) \quad \partial_t \xi(x, t) = u(\xi(x, t), t), \quad \xi(x, 0) = x.$$

For any fixed t , one can think $\xi(x, t)$ as a map of the fluid domain. By the Liouville theorem

$$\det(\nabla_x \xi(x, t)) = \det(\nabla_x \xi(x, 0)) e^{t \nabla_\xi \cdot u} = 1,$$

since

$$\nabla_\xi \cdot u = 0, \quad \det(\nabla_x \xi(x, 0)) = 1.$$

That is, ξ is a volume-preserving map in time, and any volume in the fluid domain is preserved during the fluid motion. The following is a theorem proved by Cauchy [9] [8].

THEOREM 3.1. *The vorticity along a fluid particle trajectory under the Euler dynamics, evolves according to*

$$(3.2) \quad \omega_i(\xi(x, t), t) = \xi_{i,j}(x, t) \omega_j(x, 0).$$

The equation (3.2) has the same form with the variation equation

$$\delta \xi_i(x, t) = \xi_{i,j}(x, t) \delta x_j.$$

It is not easy to extend the equation (3.2) to the viscous case, while an alternative form can be extended to the viscous case. One can introduce the tensor version of vorticity

$$\Omega_{ij} = u_{i,j} - u_{j,i}.$$

Then the theorem of Cauchy takes the following form [2]

THEOREM 3.2. *The matrix vorticity along a fluid particle trajectory under the Euler dynamics, evolves according to*

$$(3.3) \quad \xi_{m,i} \Omega_{mk}(\xi, t) \xi_{k,j} = \Omega_{ij}(x, 0).$$

Equation (3.3) is a consequence of the equation

$$\partial_t (\xi_{m,i} \Omega_{mk}(\xi, t) \xi_{k,j}) = 0.$$

The classical form of the Cauchy theorem (3.2) can be obtained from (3.3) when using the ω variable. Equation (3.3) plays a key role in proving the non-differentiability of the solution operator for Euler equations [2]. Equation (3.3) can be extended to the viscous case.

THEOREM 3.3. *The matrix vorticity along a fluid particle trajectory under the Navier-Stokes dynamics, evolves according to*

$$(3.4) \quad \partial_t (\xi_{m,i} \Omega_{mk}(\xi, t) \xi_{k,j}) = \frac{1}{Re} \xi_{m,i} \Delta \Omega_{mk}(\xi, t) \xi_{k,j}.$$

PROOF. Starting from the Navier-Stokes equations

$$\partial_t u_i + u_j u_{i,j} = -p_{,i} + \frac{1}{Re} \Delta u_i,$$

we get

$$(3.5) \quad \partial_t \Omega_{ij} + \Omega_{ik} u_{k,j} + u_{k,i} \Omega_{kj} + u_k \Omega_{ij,k} = \frac{1}{Re} \Delta \Omega_{ij}.$$

From (3.1), we have

$$(3.6) \quad \partial_t \xi_{i,j} = u_{i,k} \xi_{k,j}.$$

The material derivative of $\Omega_{ij}(\xi, t)$ is given by

$$(3.7) \quad \partial_t \Omega_{ij}(\xi, t) = \partial_t \Omega_{ij} + u_k \Omega_{ij,k}.$$

Now we can use (3.6) and (3.7) to calculate the material derivative

$$\begin{aligned} \partial_t (\xi_{m,i} \Omega_{mk}(\xi, t) \xi_{k,j}) &= \xi_{l,i} u_{m,l} \Omega_{mk} \xi_{k,j} \\ &+ \xi_{m,i} [\partial_t \Omega_{mk} + u_l \Omega_{mk,l}] \xi_{k,j} + \xi_{m,i} \Omega_{mk} u_{k,l} \xi_{l,j} \\ &= \xi_{m,i} u_{l,m} \Omega_{lk} \xi_{k,j} + \xi_{m,i} [\partial_t \Omega_{mk} + u_l \Omega_{mk,l}] \xi_{k,j} + \xi_{m,i} \Omega_{ml} u_{l,k} \xi_{k,j} \\ &= \xi_{m,i} [\partial_t \Omega_{mk} + u_l \Omega_{mk,l} + \Omega_{ml} u_{l,k} + u_{l,m} \Omega_{lk}] \xi_{k,j}, \end{aligned}$$

by rearranging the dummy indices. Using (3.5), we get (3.4). \square

4. Conclusion

By constructing explicit solutions to 2D Euler and 2D Navier-Stokes equations, we showed the rough dependence of their solution operators upon initial data. The effect of viscosity is also studied. Finally we presented an extension of a theorem of Cauchy to the viscous case.

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